Summary of Key Probability Distributions

This handout contains a summary of some important probability distributions for performance modeling of computer systems and communications networks. The distributions summarized here are uniform (continuous), uniform (discrete), binomial, Poisson, exponential, hyperexponential-2, Pareto, and bounded Pareto. Reference is made to tools on the Christensen tools page (http://www.csee.usf.edu/~christen/tools/toolpage.html) and to the Mesquite Software CSIM product.

Uniform distribution (continuous):

A random variable with uniform distribution in $a \le x \le b$ has probability density function,

$$f(x) = \begin{cases} \dfrac{1}{b-a} & a \le x \le b \\ 0 & \text{otherwise} \end{cases}$$

and a probability distribution function,

$$F(x) = \begin{cases} 0 & x < a \\ \dfrac{(x-a)}{(b-a)} & a \le x < b \\ 1 & x \ge b \end{cases}$$

The mean and variance are,

$$\mu = \frac{1}{2}(a+b) \text{ and } \sigma^2 = \frac{1}{12}(b-a)^2 .$$

To generate use `genunifc.c` from Christensen tools page or `uniform()` in the CSIM library.

Uniform distribution (discrete):

A random variable with uniform distribution in $a \le k \le b$ where $n = b - a + 1$ has a probability mass function,

$$f(k) = \begin{cases} 1/n & a \le k \le b \\ 0 & \text{otherwise} \end{cases}$$

and a cumulative distribution function,

$$F(k) = \begin{cases} 0 & k < a \\ (\lfloor k \rfloor - a + 1)/n & a \le k \le b \\ 1 & k > b \end{cases}$$

The mean and variance are,

$$\mu = \frac{1}{2}(a+b) \text{ and } \sigma^2 = \frac{1}{12}(n^2 - 1).$$

To generate use `genunifd.c` from Christensen tools page or `random_int()` in the CSIM library.

Binomial distribution (discrete):

A random variable with binomial distribution for n trials with probability p of success for each trial has probability mass function,

$$f(k) = \binom{n}{k} p^k (1-p)^{n-k} \quad 0 \le k \le n$$

The cumulative distribution function is messy.

The mean and variance are,

$$\mu = np \text{ and } \sigma^2 = np(1-p).$$

Note: Define $\lambda = np$, then as n goes to infinity the binomial distribution tends to the Poisson distribution with rate λ.

To generate use `genbin.c` from Christensen tools page or `binomial()` in the CSIM library.

Poisson distribution (discrete):

A random variable with Poisson distribution for a rate λ of arrivals has probability mass function,

$$f(k) = \frac{\lambda^k}{k!} e^{-\lambda} \quad k = 0, 1, 2, \ldots$$

The cumulative distribution function is messy.

The mean and variance are,

$$\mu = \lambda \text{ and } \sigma^2 = \lambda.$$

Note: The distribution of time between arrivals in Poisson process is exponentially distributed with mean $1/\lambda$.

To generate use `genpois.c` from Christensen tools page or `poisson()` in the CSIM library.

Exponential distribution (continuous):

A random variable with exponential distribution has density function,

$$f(t) = \begin{cases} \lambda e^{-\lambda t} & t > 0 \\ 0 & t \le 0 \end{cases}$$

and distribution function,

$$F(t) = \begin{cases} 1 - e^{-\lambda t} & x > 0 \\ 0 & x \le 0 \end{cases}$$

The mean and variance are,

$$\mu = \frac{1}{\lambda} \text{ and } \sigma^2 = \frac{1}{\lambda^2}.$$

To generate use `genexp.c` from Christensen tools page or `exponential()` in the CSIM library.

Hyperexponential-2 distribution (continuous):

A random variable with hyperexponential-2 distribution with parameters λ_1, λ_2, and p has density function,

$$f(t) = \begin{cases} p\lambda_1 e^{-\lambda_1 t} + (1-p)\lambda_2 e^{-\lambda_2 t} & t > 0 \\ 0 & t \le 0 \end{cases}$$

and distribution function,

$$F(t) = \begin{cases} 1 - pe^{-\lambda_q t} - (1-p)e^{-\lambda_2 t} & t > 0 \\ 0 & t \le 0 \end{cases}$$

The mean and variance are,

$$\mu = \frac{p}{\lambda_1} + \frac{1-p}{\lambda_2} \text{ and } \sigma^2 = 2\left(\frac{p}{\lambda_1^2} + \frac{1-p}{\lambda_2^2}\right) - \left(\frac{p}{\lambda_1} + \frac{1-p}{\lambda_2}\right)^2.$$

To generate use `genhyp1.c` or `genhyp2.c` from Christensen tools page or `hyperx()` in the CSIM library.

Pareto distribution (continuous):

A random variable with Pareto distribution with shape parameter α and minimum value k has density function,

$$f(x) = \begin{cases} \dfrac{\alpha \cdot k^\alpha}{x^{\alpha+1}} & x \ge k \\ 0 & x < k \end{cases}$$

and distribution function,

$$F(x) = \begin{cases} 1 - \left(\dfrac{k}{x}\right)^\alpha & x \ge k \\ 0 & x < k \end{cases}$$

The mean and variance are,

$$\mu = \frac{\alpha \cdot k}{\alpha - 1} \text{ and } \sigma^2 = \frac{\alpha \cdot k^2}{\left((\alpha-1)^2(\alpha-2)\right)}.$$

Note: The Pareto distribution is heavy tailed. The mean is infinity for $\alpha < 1$ and the variance is infinity for $\alpha < 2$.

To generate use `genpar1.c` from Christensen tools page or `pareto()` in the CSIM library.

Bounded Pareto distribution (continuous):

A random variable X with Bounded Pareto distribution with shape parameter α, minimum value k, and maximum value p has density function,

$$f(x) = \begin{cases} 0 & x < k \\ \dfrac{\alpha \cdot k^{\alpha}}{\left(1 - \left(\dfrac{k}{p}\right)^{\alpha}\right)} x^{-\alpha-1} & k \le x \le p \\ 0 & x > p \end{cases}$$

and distribution function,

$$F(x) = \begin{cases} 0 & x < k \\ \dfrac{p^{\alpha} \cdot \left(k^{\alpha} - x^{\alpha}\right)}{x^{\alpha} \cdot \left(k^{\alpha} - p^{\alpha}\right)} & k \le x \le p \\ 0 & x > p \end{cases}$$

The mean and variance are,

$$\mu = \frac{\alpha\left(k^{\alpha} \cdot p^{1-\alpha} - k\right)}{(\alpha - 1)\left(\left(\dfrac{k}{p}\right)^{\alpha} - 1\right)} \text{ and } \sigma^2 = \frac{\alpha \cdot \left(k^{\alpha} \cdot p^{2-\alpha} - k^2\right)}{(\alpha - 2) \cdot \left(\left(\dfrac{k}{p}\right)^{\alpha} - 1\right)} - \frac{\alpha^2 \cdot \left(k - k^{\alpha} \cdot p^{1-\alpha}\right)^2}{(\alpha - 1)^2 \cdot \left(\left(\dfrac{k}{p}\right)^{\alpha} - 1\right)^2}.$$

Note: The Bounded Pareto distribution is effectively heavy tailed, but has finite mean and variance.

To generate use `genpar2.c` from Christensen tools page (there is no CSIM library function to generate Bounded Pareto random variables).
